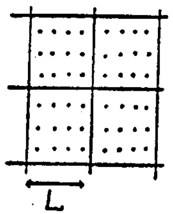
SCALING, RENORMALIZATION & CROSSOVER FOR STATIC & DYNAMICAL CRITICAL PHENOMENA IN CLASSICAL & QUANTUM SYSTEMS (M. SUZUKI)

1. DYNAMIC CELL ANALYSIS ⟶ SCALING LAW
2. LINEAR & NONLINEAR DYNAMICAL SCALING RELATIONS IN R.G. THEORY
3. DERIVATION OF FISHER'S FINITE-SIZE SCALING LAW USING R.G. THEORY
4. DYNAMIC FINITE-SIZE SCALING LAW-CROSSOVER EFFECT IN DYNAMICS
5. RELATION BETWEEN QUANTUM SPIN SYSTEM & ISING SYSTEM — QUANTUM-CLASSICAL CROSSOVER —

(Ref. M. Suzuki : Prog. Theor. Phys. 51 (1974) 1257.)

Dynamic cell analysis ⟶ scaling law



TDGL model

fluctuation

$$\frac{\partial S}{\partial t} = -\left(\frac{\Gamma}{k_B T}\right)\frac{\delta \mathcal{H}}{\delta S} + \eta(r,t)$$

$$\langle \eta(r,t)\eta(r',t')\rangle = 2\Gamma\,\delta(r-r')\,\delta(t-t')$$

scale trans. $\varepsilon = (T-T_c)/T_c$, $h = \mu_B H/k_B T$

$$R \longrightarrow R' = R/L,\quad \varepsilon \rightarrow \varepsilon' = \varepsilon L^{y},\quad h \rightarrow h' = hL^{x}$$

$$S \rightarrow S' = L^{d-x} S \quad ; \quad x = d - \beta/\nu$$

Coarse graining of time (Markoffian appr.) $\left\}\begin{array}{l} t \longrightarrow t' = L^{-z} t \\ \Gamma \longrightarrow \Gamma' = L^{\phi}\Gamma \end{array}\right.$

$\Downarrow$

new eq. of motion is invariant

$$\frac{\partial S'}{\partial t'} = -\left(\frac{\Gamma'}{k_B T'}\right)\frac{\delta \mathcal{H}'}{\delta S'} + \eta' \; ; \qquad \eta' = \eta L^{x+\phi}$$

if $z = 2x - d + \phi = \frac{\gamma}{\nu} + \phi = 2 - \eta + \phi$

Note that $(\mathcal{H}/k_B T) = L^{-d}(\mathcal{H}'/k_B T')$

and $[l.h.] = L^{d-x}L^{z}$, $[r.h.] = L^{\phi}L^{-d}L^{-(d-x)}$

invariance of eq. of motion for scale trans.

$\Downarrow$

time corr. func. $S(R,\varepsilon,h:\omega) = \int_0^\infty \langle S(0,0)S(R,t)\rangle e^{-i\omega t}\,dt$

$$S(R,\varepsilon,h;\omega) = \underbrace{L^{2(x-d)+z}}\, S(R/L,\, L^{y}\varepsilon,\, hL^{x},\, L^{z}\omega)$$

$\Uparrow$

static matching $\int S(R,\varepsilon,h)\,d\omega \propto R^{-(d-2+\eta)}$

Solution ⟶ dynamic scaling

$$S(R,\varepsilon,h:\omega) = R^{2(x-d)+z} S(R,\, \varepsilon^{1/y},\, \varepsilon h^{-y/x};\, \omega R^{z})$$

where $y = \frac{1}{\nu}$ and $z = 2 - \eta + \phi$

Linear and Nonlinear Dynamic Scaling Relations in the Renormalization Group Theory

→ Proof of Existence → Prog. Theor. Phys. 53('75)1657; 55('76)383, 1064

Generating function: (M.S. Phys. Letters 58A (1976) 435)

$$\Phi(\lambda,h,\varepsilon,t) = \frac{1}{\Omega}\log\int\prod_{k<1} d\sigma_k\, e^{\lambda M} e^{t\Gamma} e^{hM-\mathcal{H}}$$

$$\mathcal{H}_\ell = \sum_{n=1}^{\infty}\Omega^{1-n}\sum_{k_1,k_2\cdots k_{2n}<1} u_{2n,\ell}(k_1,\cdots,k_{2n})\,\sigma_{\ell k_1}\cdots\sigma_{\ell k_{2n}}$$

$$\Gamma_\ell = \int\prod_{q<1} d\sigma_{\ell q}\,\frac{\partial}{\partial\sigma_{\ell q}}\left(\frac{\partial\mathcal{H}_\ell}{\partial\sigma_{\ell,-q}} + \frac{\partial}{\partial\sigma_{\ell,-q}}\right) + \cdots\cdots$$

to rescale the variables as

$$k\to bk,\quad \Omega\to b^d\Omega,\quad \sigma_{\ell k}\to b^{1-\eta/2}\sigma_{\ell+1,bk},\quad t\to b^{-z}t$$

$$t_\ell\Gamma_\ell = t_{\ell+1}\Gamma_{\ell+1} = \text{invariant}$$

(confirmed perturbationally c.f. M.S. & F.T.) (Prog. Theor. Phys. 52('74)722)

Note that $\lambda M = \lambda\Omega^{1/2}\sigma_{0,0}$ and $hM = h\Omega^{1/2}\sigma_{0}$.

renormalization proc. $(b^{-1}<k<1)$

→ does not change results

⇓ with $\lambda_{\ell+1} = b^x\lambda_\ell$, $h_{\ell+1} = b^x h_\ell$,

we have

$$\Phi_\ell(\lambda,h,\varepsilon,t) = b^{-d}\Phi_{\ell+1}(\lambda,h,\varepsilon,t) + f_\ell(\varepsilon,t)$$

$$\Phi_{\ell+1}(\lambda,h,\varepsilon,t) = \Phi_\ell(\lambda b^x, hb^x, \varepsilon b^y, tb^{-z}) \quad \{u_{2n,\ell}\}$$

$$\Phi_\ell(\lambda,h,\varepsilon,t) = b^{-d}\Phi_\ell(\lambda b^x, hb^x, \varepsilon b^y, tb^{-z}) + f_\ell(\varepsilon,t)$$

Differentiate w.r.to λ and define $m_\ell \equiv \left(\frac{\partial\Phi_\ell}{\partial\lambda}\right)_{\lambda=0}$

$$\therefore\ m_\ell(h,\varepsilon,t) = b^{x-d} m_\ell(hb^x, \varepsilon b^y, tb^{-z})$$

General sol.

$$m(t) = h^{1/\delta}F_1(h\varepsilon^{-\beta\delta}, t\varepsilon^{\nu z}) \quad\text{or}\quad m(t) = \varepsilon^\beta F_2(h\varepsilon^{-\beta\delta}, t\varepsilon^{\nu z})$$

where

$$\nu = \frac{1}{y},\quad \delta = \frac{x}{d-x} = \frac{d+2-\eta}{d-2+\eta},\quad \beta = \frac{x}{\delta y} = \frac{\gamma}{\delta-1}$$

Note that $m(0) = h^{1/\delta}F_1(0,0)$

$$m(t) = m(0)\, f_1\!\left(\frac{m(0)}{\varepsilon^\beta}, t\varepsilon^{\nu z}\right)$$

$$\text{or}\quad m(t) = \varepsilon^\beta f_2\!\left(\frac{\varepsilon^\beta}{m(0)}, t\varepsilon^{\nu z}\right)$$

(i) linear relaxation $m(0) \ll \varepsilon^\beta$

$$\tau^{(\ell)} = \int_0^\infty f_1(0, t\varepsilon^{\nu z})\,dt \propto \varepsilon^{-\nu z} \propto \varepsilon^{-\Delta^{(\ell)}}$$

(ii) nonlinear relax. $\varepsilon^\beta \ll m(0)$

$$\tau^{(n\ell)} = \varepsilon^\beta\int_0^\infty f_2(0, t\varepsilon^{\nu z})\,dt \propto \varepsilon^{\beta-\nu z} \propto \varepsilon^{-\Delta^{(n\ell)}}$$

$$\therefore\ \Delta^{(n\ell)} = \Delta^{(\ell)} - \beta = \frac{1}{2} + \frac{1}{3}(4-d) + \cdots \quad \left(\begin{matrix}\text{Fisher}\\ \text{Rácz}\end{matrix}\right)$$

Crossover from lin. to nonlin. at $m(0) = m^x = \varepsilon^\beta$

Derivation of Fisher's Finite-Size Scaling Law on the basis of the RG theory

a) type A $\quad L \times L \times \cdots \times L \;; \quad \varepsilon = (T - T_c(\infty))/T_c(\infty)$

Consider $\Phi_\ell(\varepsilon, k_0, h) = \frac{1}{\Omega} \log \int \prod_{k_0<k<1} d\sigma_{\ell k}\, e^{hM - \mathcal{H}_\ell}$

$\boxed{k_0 = L^{-1}}$ (no constraint)

to rescale the variables as

$$k \to bk, \quad k_0 \to bk_0, \quad h \to b^x h, \quad \varepsilon \to b^y \varepsilon$$

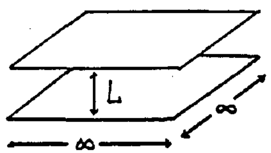
$$\begin{cases} \Phi_\ell(h, \varepsilon, k_0) = b^{-d} \Phi_{\ell+1}(h, \varepsilon, k_0) + f_\ell(\varepsilon), \\ \Phi_{\ell+1}(h, \varepsilon, k_0) = \Phi_\ell(hb^x, \varepsilon b^y, k_0 b) \end{cases}$$

$$M_\ell \equiv \frac{\partial \Phi_\ell}{\partial h} \rightarrow M_\ell(h, \varepsilon, k_0) = b^{x-d} M_\ell(hb^x, \varepsilon b^y, k_0 b)$$

General sol. $M(h, \varepsilon, L) = L^{-\beta/\nu} m(h\varepsilon^{-\beta\delta}, \varepsilon L^{1/\nu})$

(Fisher's Finite-size scaling law)

b) type B $\quad \infty \times \infty \times L \;;$



$$\begin{cases} \varepsilon = (T - T_c(\infty))/T_c(\infty) \\ \dot{\varepsilon} = (T - T_c(L))/T_c(\infty) \end{cases}$$

③

Consider $\Phi_\ell(h, \varepsilon, k_0) = \frac{1}{\Omega} \log \int \prod_{\substack{0 \le k_x, k_y < 1 \\ k_0 < k_z < 1}} d\sigma_{\ell, k}\, e^{-\mathcal{H}_\ell + hM}$

we have

$$\Phi_\ell(h, \dot{\varepsilon}, k_0) = b^{-d} \Phi_\ell(h, \dot{\varepsilon}, k_0) + f_\ell(\dot{\varepsilon})$$

Similarly to Type A, we get

$$M_\ell(h, \dot{\varepsilon}, k_0) = b^{x-d} M_\ell(hb^x, \dot{\varepsilon} b^y, k_0 b)$$

$$\therefore M(h, \dot{\varepsilon}, k_0) = L^{-\beta/\nu} m_2(h\dot{\varepsilon}^{-\beta\delta}, \dot{\varepsilon} L^{1/\nu})$$

(for no constraint)

Dynamic Finite-Size Scaling Law

— Crossover Effect in Dynamics

For simplicity, consider Type B

$$\Phi_\ell(\lambda, h, \dot{\varepsilon}, k_0, t) = \frac{1}{\Omega} \log \int \prod_{\substack{0<k_x,\, k_y<1 \\ k_0<k_z<1}} d\sigma_{\ell,k}\, e^{\lambda M} e^{t\Gamma_\ell} e^{hM-\mathcal{H}_\ell}$$

to rescale the variables as

$$k \to bk, \quad k_0 \to bk_0, \quad t \to b^{-z}t$$

$$\Phi_\ell(\lambda, h, \dot{\varepsilon}, k_0, t) = b^{-d}\Phi_\ell(\lambda b^x, hb^x, \dot{\varepsilon}b^y, k_0 b, tb^{-z}) + f_\ell(\dot{\varepsilon}t)$$

$$\therefore \underline{M_\ell(h, \dot{\varepsilon}, k_0, t) = b^{x-d} M_\ell(hb^x, \dot{\varepsilon}b^y, k_0 b, tb^{-z})}$$

General sol.

$$M(h, \varepsilon, L, t) = L^{-\beta/\nu} f_1(h\dot{\varepsilon}^{-\beta\delta}, \dot{\varepsilon}L^{1/\nu}, tL^{-z})$$

Dynamic Finite-Size Scaling Law

◎ $M(t) = t^{-\psi} m(Lt^{-\frac{1}{z}})$ at $\dot{\varepsilon} = 0$; $\boxed{\psi = \frac{\beta}{z\nu}}$

◎ $M(t) \propto t^{-\dot{\psi}}$ for $t \to \infty$, $L < \infty$; $\boxed{\dot{\psi} = \frac{\dot{\beta}}{z\dot{\nu}}}$

Then, $m(\xi) \approx M_0 \xi^{-z(\psi - \dot{\psi})}$ as $\xi \to 0$

($m(\xi)$ = constant as $\xi \to \infty$)

Thus, $M(t) \approx M_0 L^{-z(\psi-\dot{\psi})} t^{-\dot{\psi}}$

$$M(t) \approx M_0 L^{-z\psi} (t/L^z)^{-\psi}$$

(i) $t > t^\times \implies M(t) \sim t^{-\dot{\psi}}$ ($d=2$)

(ii) $t < t^\times \implies M(t) \sim t^{-\psi}$ ($d=3$)

where $t^\times = L^z$ (Crossover time)

or $\omega \gtrless \omega^\times$; $\omega^\times = L^{-z} = k_0^z$

Crossover effect in Critical Dynamics

Relationship between Quantal Spin Systems and Ising Systems — Quantum-Classical Crossover

(M. Suzuki, Prog. Theor. Phys. 56 (1976) No.5)

Ex. $\mathcal{H}_Q^{(d)} = -\sum \sigma_i^z \sigma_j^z - \Gamma \sum \sigma_j^x$

$\boxed{\text{Th. 1}}$ $\mathcal{H}_Q^{(d)}$ at $T=0$ $\overset{\text{equivalent}}{\Longleftrightarrow}$ $\mathcal{H}_{Ising}^{(d+1)}$

Proof Put $n = \Gamma/kT$ and note that

$$\mathcal{F} = -kT \log \mathrm{Tr} \exp(-\mathcal{H}_Q^{(d)}/kT)$$

$$= -\frac{\Gamma}{n} \lim_{m\to\infty} \mathrm{Tr}\left[\exp\left(\sum \frac{J_{ij}}{m\Gamma}\sigma_i^z\sigma_j^z\right)\exp\left(\frac{1}{m}\sum\sigma_j^x\right)\right]^{mn}$$

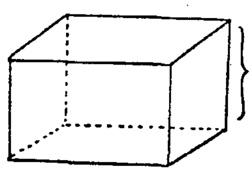
Trotter's formula

$$e^{A+B} = \lim_{m\to\infty}\left(e^{\frac{A}{m}} e^{\frac{B}{m}}\right)^m$$

Proof → Generalized T.F. M. Suzuki: Commun. Math. Phys. 51 (1976) 183

$$\mathcal{F} = -\frac{\Gamma}{n}\lim_{m\to\infty}\log\left[\left(\frac{1}{2}\sinh\frac{2}{m}\right)^{\frac{Nmn}{2}}\sum_{\sigma_{ij}=\pm 1}\exp\mathcal{H}_{eff}^{(n,m)}\right]$$

$$\mathcal{H}_{eff}^{(n,m)} = \frac{1}{m\Gamma}\sum_{ij\in R^d}\sum_{k=1}^{mn} J_{ij}\sigma_{i,k}\sigma_{j,k} + \frac{1}{2}\left(\log\coth\frac{1}{m}\right)\sum\sum\sigma_{i,k}\sigma_{i,k+1}$$



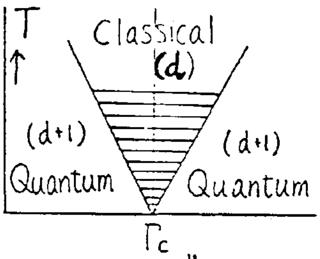
$\} \; mn \propto \frac{1}{T}$

Real space (d dimensions)

Corollary

Monte-Carlo Calculations of Quantum Spin Systems. (Suzuki, Miyashita, Kuroda)

$\boxed{\text{Th.2}}$ Quantum - Classical Crossover



Fisher's Finite-Size Scaling

⇓

Size $mn \propto \frac{1}{T}$

↕

Inverse temperature

⇓

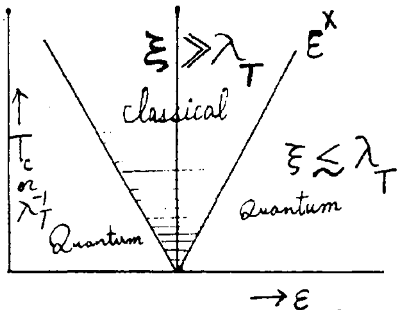
Quantum Crossover Effect in the Gas-Liquid P.T.

(M. Suzuki, Prog. Theor. Phys. 56 (1976) No.3)

crossover at $\xi \simeq \lambda_T \equiv \hbar(2mkT)^{-\frac{1}{2}}$

He^4, He^3 .... $\gamma = 1.13$ ($d+1 = 4$ dimensions)

$\longleftrightarrow$ Xe, CO_2 .... $\gamma = 1.25$



$\lambda_T = \frac{\hbar}{\sqrt{2mkT}}$ (thermal de Broglie wave length)

$\xi = \xi_0 \varepsilon^{-\nu}$ (correlation length)

$$\varepsilon^x \simeq \left(\frac{2mkT_c\,\xi_0^2}{\hbar^2}\right)^{-\frac{1}{2\nu}} \equiv (\lambda^*)^{-\frac{1}{\nu}} \quad (\lambda^* = \text{de Boer's para})$$

数値計算

R.J. Elliott et al. J. Phys. C: Solid St. Phys. 4 ('71) 2359.

A. Yanase et al. J. Phys. Soc. Japan (in press)

J. Oitmaa et al. J. Phys. C: Solid St. Phys. 9 ('76) 2093.

Applications

Z. Friedman: Phys. Rev. Lett. 36 ('76) 1326.

K. Subbarao: two preprints.